with the condition

$$\lambda_1 + \mu_1 + \nu_1 = 0, \tag{3}$$

and that of a second plane embracing (1) is

$$\frac{(x-a)\lambda_2}{l} + \frac{(y-b)\mu_2}{m} + \frac{(z-c)\nu_2}{n} = 0, \tag{4}$$

with the condition

$$\lambda_2 + \mu_2 + \nu_2 = 0. \tag{5}$$

If (2) and (4) are orthogonal,

$$\frac{\lambda_1}{l} \cdot \frac{\lambda_2}{l} + \frac{\mu_1}{m} \cdot \frac{\mu_2}{m} + \frac{\nu_1}{n} \cdot \frac{\nu_2}{n} = 0, \tag{6}$$

and if (2) and (4) touch

$$a_1x^2 + b_1y^2 + c_1z^2 = 1, \tag{7}$$

$$\left(\frac{\lambda_1}{l}\right)^2\Big/a_1 + \left(\frac{\mu_1}{m}\right)^2\Big/b_1 + \left(\frac{\nu_1}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_1}{l} + \frac{b\mu_1}{m} + \frac{c\nu_1}{n}\right)^2, \tag{8}$$

and

$$\left(\frac{\lambda_2}{l}\right)^2\Big/a_1 + \left(\frac{\mu_2}{m}\right)^2\Big/b_1 + \left(\frac{\nu_2}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_2}{l} + \frac{b\mu_2}{m} + \frac{c\nu_2}{n}\right)^2. \tag{9}$$

Now (3), (5), (6), (8), (9) is a system of *five* equations for the determination of the *six* ratios λ_1/ν_1, μ_1/ν_1; λ_2/ν_2, μ_2/ν_2; and l/n, m/n, giving an indeterminate solution. The values of l/n, m/n are the only ones needed in (1).

There seems to be a missing condition in the statement of the problem.

If the direction of the line (1) were constant, or $l : m : n$, constant, the *locus* of the line (1) would be a right circular cylinder, or, in other words, the locus of the line of intersection of constant direction of pairs of orthogonal tangent planes to a central conicoid is a right circular cylinder.

Note.—The single condition on the line restricts it to be a member of a line complex whose order apparently may be as high as 8. It would be desirable to determine this explicitly. EDITORS.

497. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two spheres on any line passing through any point common to the two spheres.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let the plane of the common circle of the spheres be chosen for yz-plane, and the line of centers for x-axis. Let the radius of the common circle be k, and let $(a, 0, 0)$ and $(b, 0, 0)$ be the centers of the two spheres. Then the equation of one sphere is

$$(x-a)^2 + y^2 + z^2 = a^2 + k^2,$$

or

$$x^2 + y^2 + z^2 - 2ax = k^2; \tag{1}$$

and, similarly, the equation of the other is

$$x^2 + y^2 + z^2 - 2bx = k^2. \tag{2}$$

Let l, m, n be the direction cosines of an arbitrary line through the point M $(0, 0, k)$. Then the equation of the line may be written

$$\frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} = r, \tag{3}$$

where r is the length of the segment joining $(0, 0, k)$ and (x, y, z).

To find the length of the segment MP cut from this line by the sphere (1), we substitute $x = lr$, $y = mr$, $z = nr + k$ in equation (1) and solve for that value of r which is not zero. We thus find at once

$$r = MP = 2al - 2kn.$$

Substituting the same values in (2), we find likewise for the segment MQ intercepted by the

second sphere,

$$MQ = 2bl - 2kn.$$

If R be the mid-point of PQ, then

$$MR = \frac{MP + MQ}{2} = (a + b)l - 2kn.$$

Hence, if in (3) we set

$$r = (a + b)l - 2kn, \tag{4}$$

we readily obtain expressions for the coördinates x, y, z of the middle point R of the segment determined by the two spheres on the line through $(0, 0, k)$ and having the direction (l, m, n). To find the equation of the locus of R for all directions of the line we must eliminate l, m, n, which we do as follows:

From (3), we have,

$$x^2 + y^2 + (z - k)^2 = r^2; \tag{5}$$

and from (4) and (3),

$$r = (a + b)\left(\frac{x}{r}\right) - 2k\left(\frac{z - k}{r}\right),$$

or

$$r^2 = (a + b)x - 2k(z - k). \tag{6}$$

Equating the values of r^2 given by (5) and (6), we have for the equation of the locus of R,

$$x^2 + y^2 + z^2 - (a + b)x = k^2. \tag{7}$$

The locus is, therefore, a sphere whose center is midway between the centers of the given spheres and which contains all points common to these spheres. It is evident then that the choice of any point other than M common to the two spheres would lead to the same locus.

Also solved by the Proposer.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, Washington, D. C.

Determine a function of x independent of b, such that

$$\int_b^{b+1} f(x)dx = \frac{1}{b+1},$$

the real part of b being positive.

Solution by C. F. Gummer, Kingston, Ontario.

There is one solution only satisfying the conditions:
(i) $f(x)$ is one-valued and continuous in the right half plane of complex numbers, and
(ii) $f(x + n)$ approaches 0 as n approaches $+\infty$ through integral values for every x.
Let $f(x)$ be such a solution. Differentiating the equation

$$\int_b^{b+1} f(x)dx = \frac{1}{b+1}, \tag{1}$$

we get, after substituting x for b,

$$f(x + 1) - f(x) = -\frac{1}{(x + 1)^2}.$$

Similarly,

$$\begin{array}{lcl} f(x+2) - f(x+1) & = & -\dfrac{1}{(x+2)^2}, \\ \vdots & & \vdots \\ f(x+n) - f(x+n-1) & = & -\dfrac{1}{(x+n)^2}. \end{array}$$

Adding these equations, we have